Math 1433

13 November 2023

Warm-up: Multiply $\begin{bmatrix} 2 & 3 \\ -1 & 5 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix}$.

Rule: if $M = AB$ then m_{ij} = (row i of A) · (column j of B).

Example: $\begin{bmatrix} 2 & 3 \\ -1 & 5 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 11 & 8 \\ 1 & 9 \end{bmatrix}$

Example: $\begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ -1 & 5 \end{bmatrix} = \begin{bmatrix} 7 & 17 \\ 0 & 13 \end{bmatrix}$

Note: when multiplying matrices, AB and BA can be different!

Last time

Rule: if $M = AB$ then m_{ij} = (row i of A) · (column j of B).

Example: $$\underbrace{\begin{bmatrix} 4 & 1 & 0 \\ -2 & 1 & 5 \end{bmatrix}}_{2 \times 3} \underbrace{\begin{bmatrix} 1 & -1 \\ 2 & 9 \\ 3 & 1 \end{bmatrix}}_{3 \times 2} = \underbrace{\begin{bmatrix} 6 & 5 \\ 15 & 16 \end{bmatrix}}_{2 \times 2}$$

The "inner" numbers must agree for AB to exist.

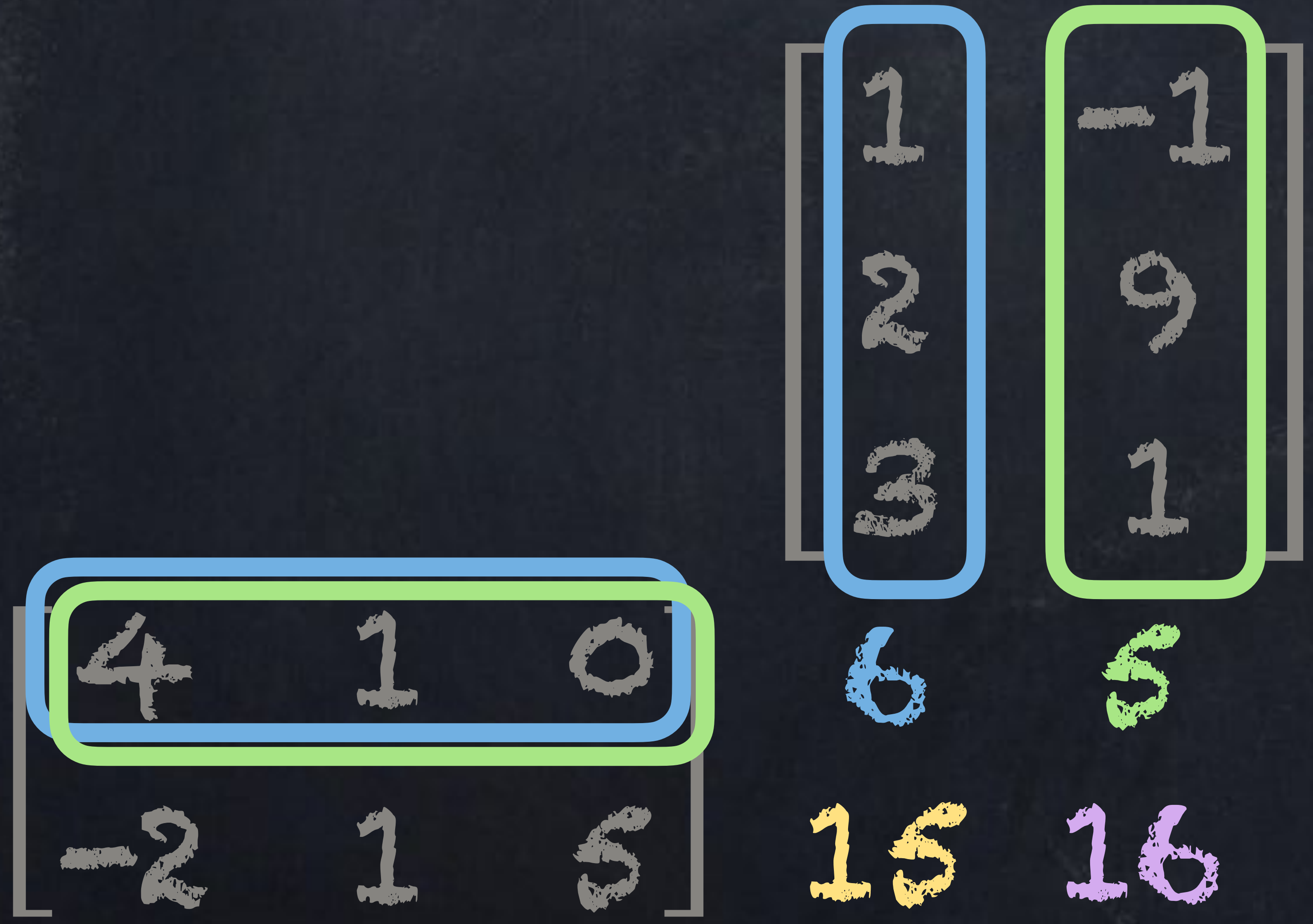
The "outer" numbers give the dimensions of AB.

Last time

Rule: if $M = AB$ then m_{ij} = (row i of A) · (column j of B).

dot product of two vectors

Example: $$\begin{bmatrix}4 & 1 & 0\\ -2 & 1 & 5\end{bmatrix}\begin{bmatrix}1 & -1\\ 2 & 9\\ 3 & 1\end{bmatrix} = \begin{bmatrix}6 & 5\\ 15 & 16\end{bmatrix}$$



$$[4,1,0]\cdot[1,2,3] = 4+2+0 = 6$$

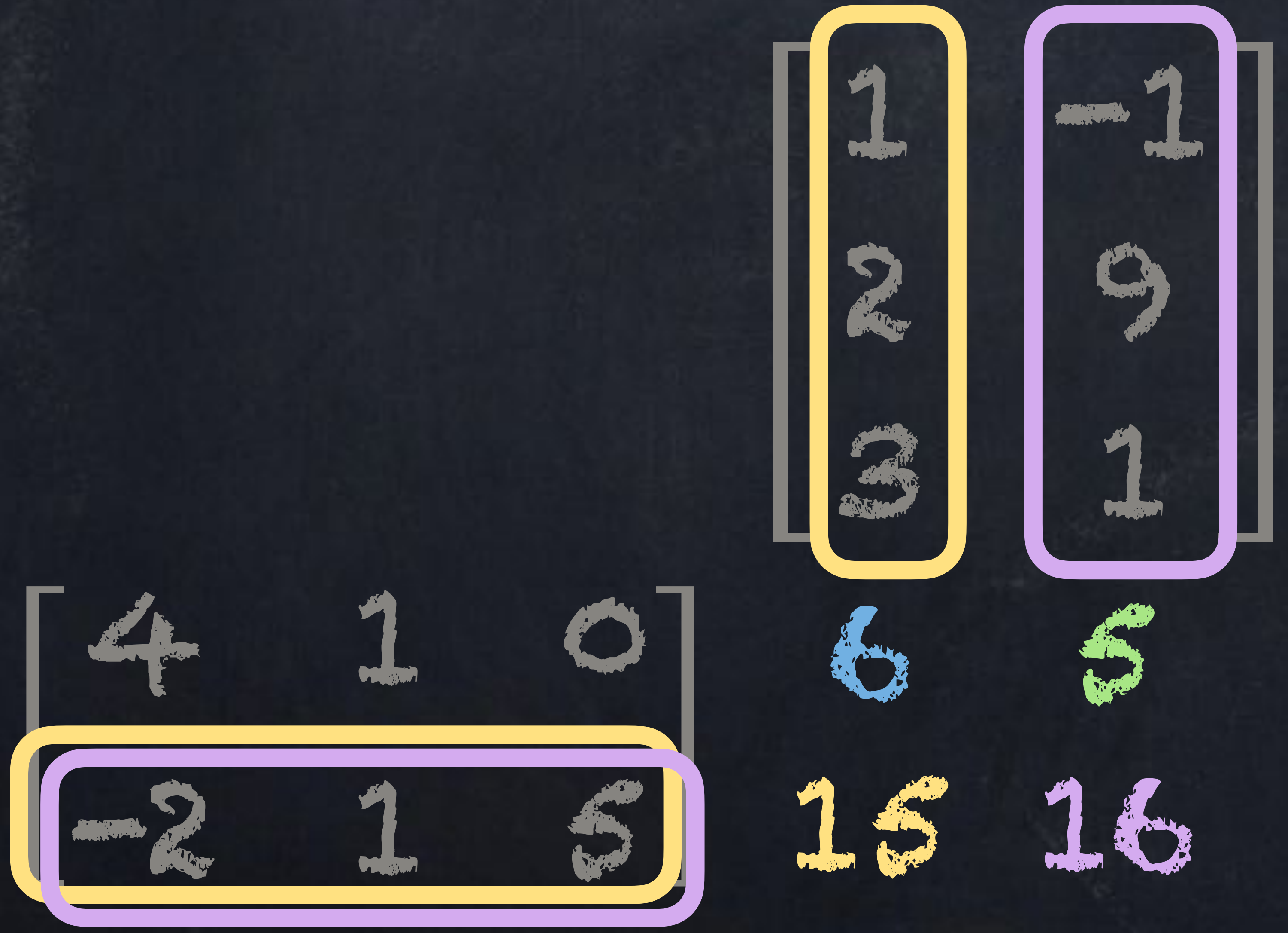
$$[4,1,0]\cdot[-1,9,1] = -4+9+0 = 5$$

Last time

Rule: if $M = AB$ then m_{ij} = (row i of A) · (column j of B).

dot product of two vectors

Example: $$\begin{bmatrix} 4 & 1 & 0 \\ -2 & 1 & 5 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & 9 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 5 \\ 15 & 16 \end{bmatrix}$$



$[-2,1,5] \cdot [1,2,3] = -2+2+15 = 15$

$[-2,1,5] \cdot [-1,9,1] = 2+9+5 = 16$

Rule: if $M = AB$ then m_{ij} = (row i of A) · (column j of B).

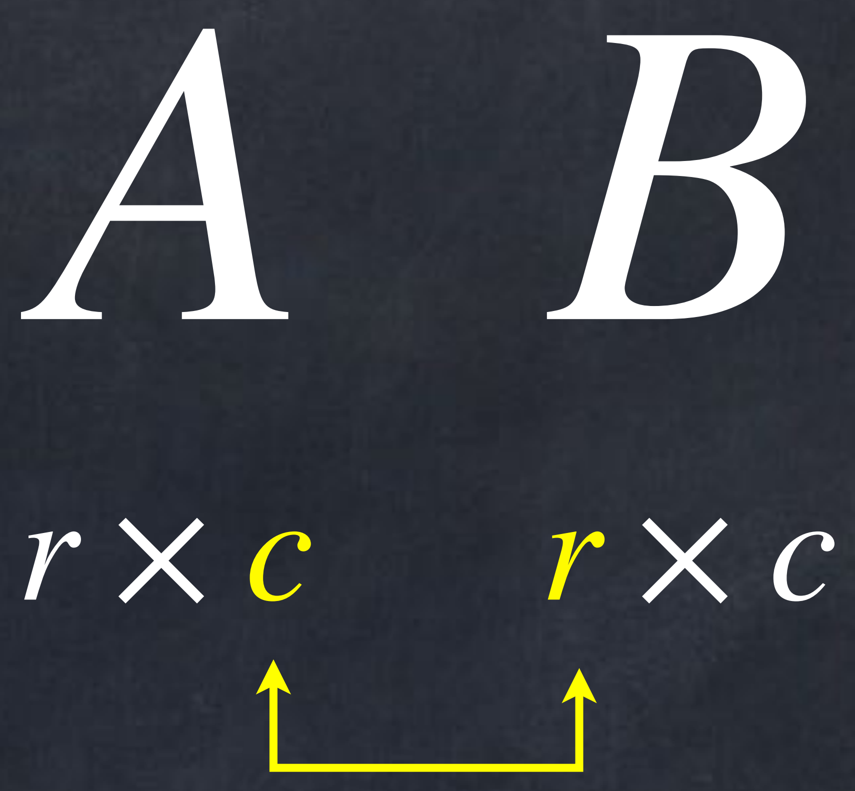
Example: $$\underset{\underline{2} \times 3}{\begin{bmatrix} 4 & 1 & 0 \\ -2 & 1 & 5 \end{bmatrix}} \underset{3 \times \underline{2}}{\begin{bmatrix} 1 & -1 \\ 2 & 9 \\ 3 & 1 \end{bmatrix}} = \underset{\underline{2 \times 2}}{\begin{bmatrix} 6 & 5 \\ 15 & 16 \end{bmatrix}}$$

Example: $$\underset{\underline{3} \times 2}{\begin{bmatrix} 1 & -1 \\ 2 & 9 \\ 3 & 1 \end{bmatrix}} \underset{2 \times \underline{3}}{\begin{bmatrix} 4 & 1 & 0 \\ -2 & 1 & 5 \end{bmatrix}} = \underset{\underline{3 \times 3}}{\begin{bmatrix} 6 & 0 & -5 \\ -10 & 11 & 45 \\ 10 & 4 & 5 \end{bmatrix}}$$

Last time

Warnings

🌟 The matrix multiplication AB is only possible if…



$$\begin{array}{cc} A & B \\ r \times c & r \times c \end{array}$$

…these are equal.

If they are, the “outer” dimensions become the dimensions of matrix AB.

🔄 In general, AB and BA do not have to be equal (in fact, one might exist and the other might not exist!).

If A is a 2×3 matrix and B is a 5×3 matrix,

- does $A + B$ exist?
- does $B + A$ exist?
- does $A + A$ exist?
- does AB exist?
- does BA exist?
- does AA exist?

If C is a 3×3 matrix,

- does AC exist?
- does CB exist?
- does CC exist?

Algebra with matrices

Matrices follow all the usual algebra rules *except* that AB is not necessarily the same as BA.

If the left and right sides of the equations both exist, then these are true:

- $A + B = B + A$
- $A(BC) = (AB)C$ ← Because of this, we can just write ABC without parentheses.
- $A(B + C) = AB + AC$
- $(A + B)(C + D) = AC + BC + AD + BD$
- $(A + B)^2 = A^2 + AB + BA + B^2$ (note: not …$2AB$…)

Previously

The symbol $\mathbb{R}$ is used for the collection of all real numbers.

The collection of all 2D vectors is $\mathbb{R}^2$.

The collection of all 3D vectors is $\mathbb{R}^3$, and so on.

The functions you study in school and in Analysis 1 are usually from $\mathbb{R}$ to $\mathbb{R}$, meaning the input and output are numbers.

An example of a function from $\mathbb{R}^2$ to $\mathbb{R}^2$ could be $f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x-y \\ e^x \end{bmatrix}$, which

can also be written as

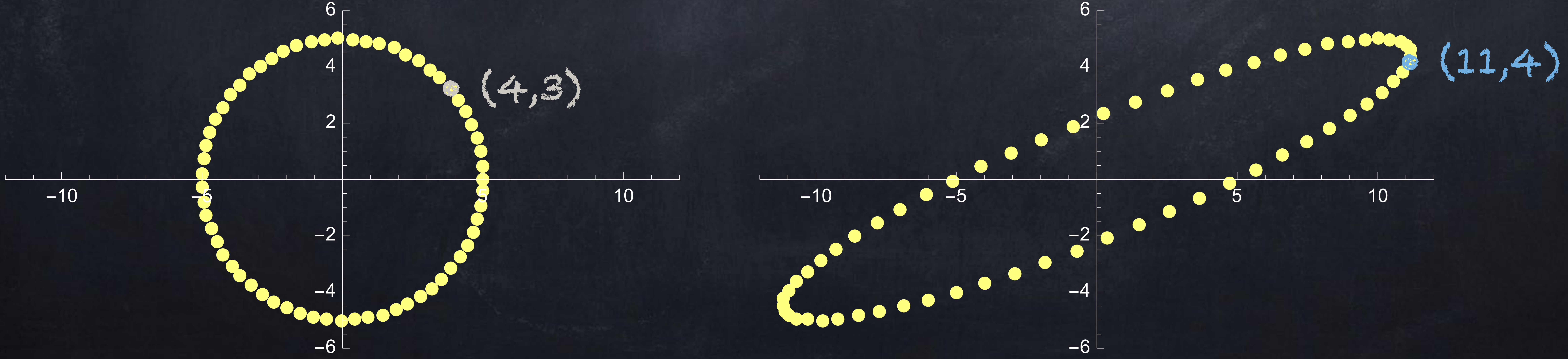
$$f(x\hat{\imath} + y\hat{\jmath}) = (x-y)\hat{\imath} + e^x\,\hat{\jmath} \qquad\qquad f(x, y) = \left(x-y, e^x\right)$$

Transformations

The **image** of a point or set of points under a transformation is the output of the function for that input.

Example: for $f(x, y) = (2x + y, x)$,

- the image of the point $(4, 3)$ is the point $(11, 4)$.
- the image of the circle $x^2 + y^2 = 25$ is a rotated ellipse.



Linear transformations

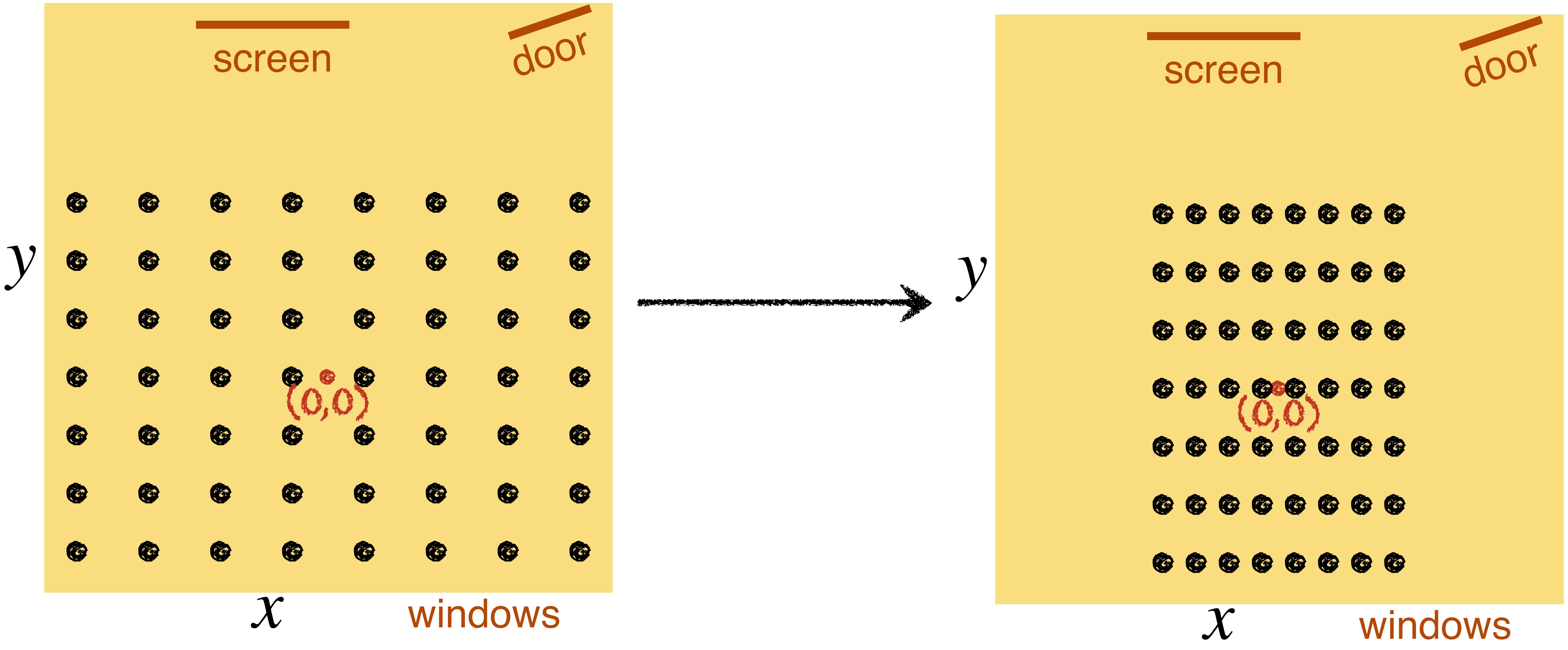
A **linear transformation** from $\mathbb{R}^2$ to $\mathbb{R}^2$ is a function with the following two properties:

- the image of a line must be a line or point,
- the origin should not move.

What other kinds of pictures can this create?

What does this look like in terms of formulas?

$$f(x, y) = (\frac{1}{2}x, y)$$



A linear transformation $\mathbb{R}^2 \to \mathbb{R}^2$ can

- rotate

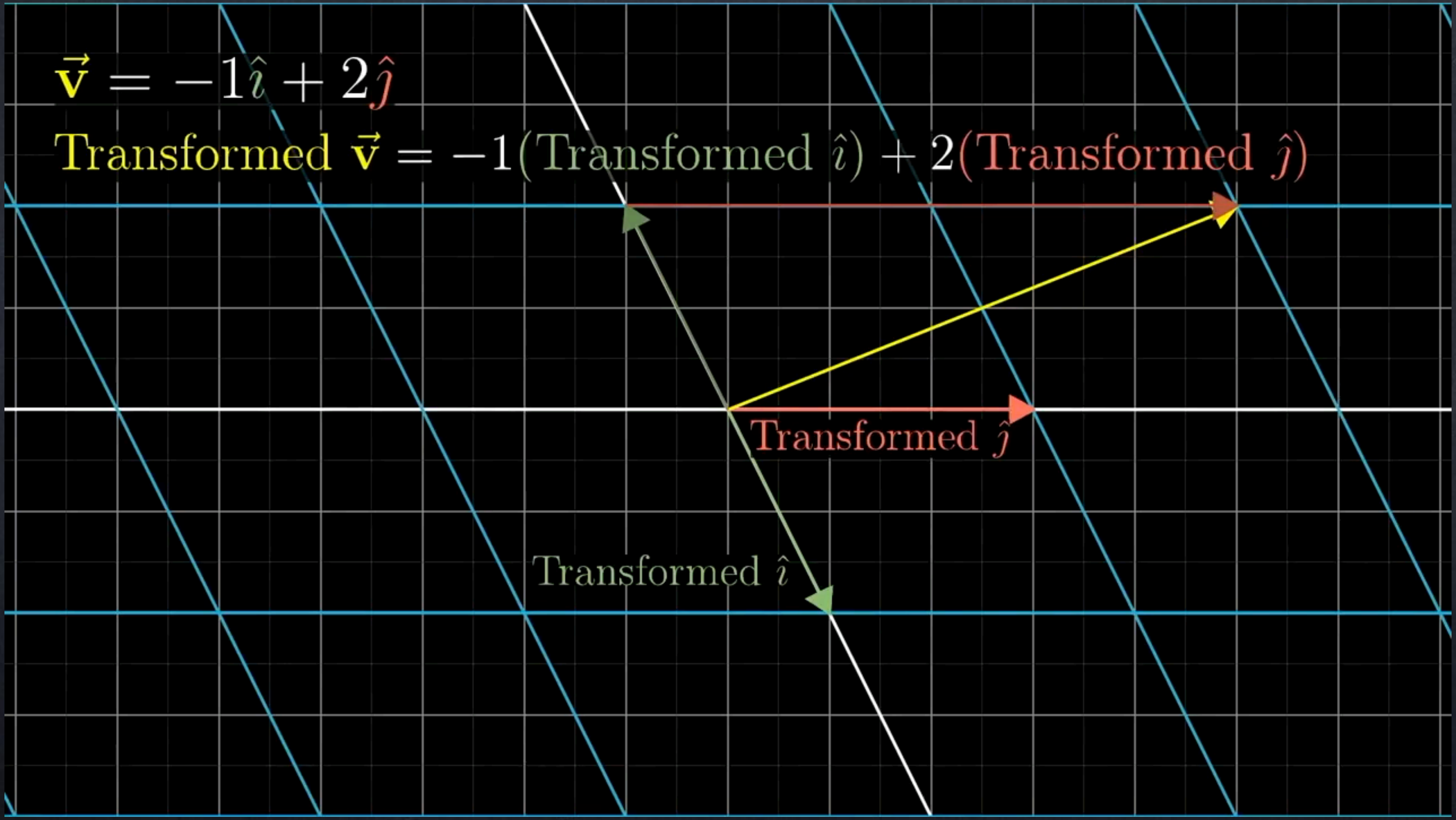
- reflect

- shear

- scale (including disproportionately)

or do any combination of these to the original shape .

From 3B1B’s “Linear transformations and matrices”
youtube.com/watch?v=kYB8IZa5AuE



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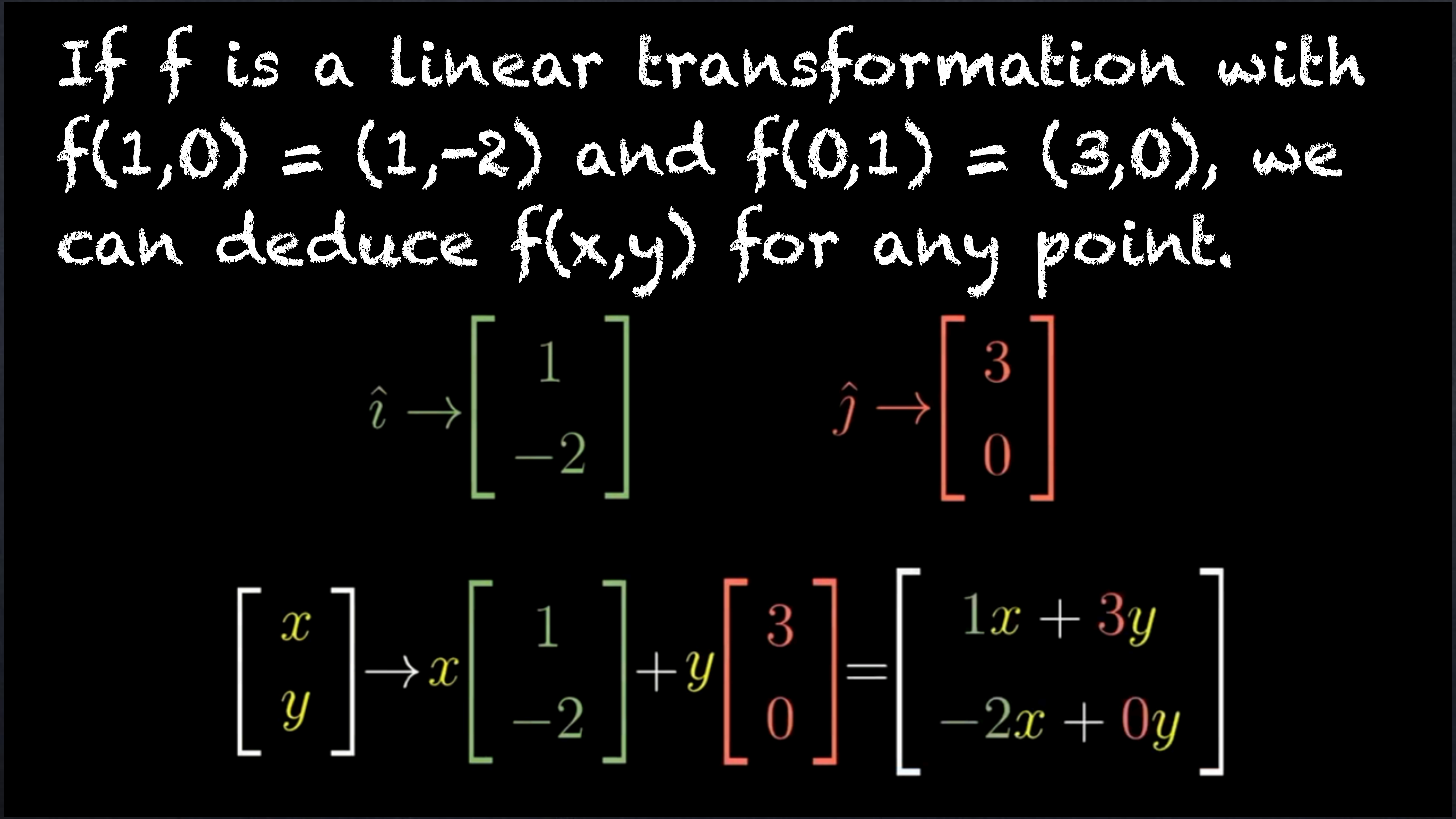

If f is a linear transformation with f(1,0) = (1,-2) and f(0,1) = (3,0), we can deduce f(x,y) for any point.

$$\hat{\imath} \rightarrow \begin{bmatrix} 1 \\ -2 \end{bmatrix} \qquad \hat{\jmath} \rightarrow \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} \rightarrow x\begin{bmatrix} 1 \\ -2 \end{bmatrix} + y\begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 1x + 3y \\ -2x + 0y \end{bmatrix}$$

Linear transformations

By thinking carefully about grid lines, we can see that

$$f(x\hat{\imath} + y\hat{\jmath}) = xf(\hat{\imath}) + yf(\hat{\jmath})$$

for any linear transformation function $f : \mathbb{R}^2 \to \mathbb{R}^2$.

- The examples of rotation, shearing, etc., do have this properties.

Officially, $f : \mathbb{R}^n \to \mathbb{R}^m$ is a **linear transformation** if

$$f(a\vec{v} + b\vec{w}) = af(\vec{v}) + bf(\vec{w})$$

for all $a, b, \vec{v}, \vec{w}$. Note $\mathbb{R}^n$ and $\mathbb{R}^m$ can be any dimensions here.

Suppose $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation for which $f(1,0) = (3, -2)$ and $f(0, 1) = (2, 1)$. Calculate $f(5, 7)$.

We can also write $f(\hat{\imath}) = 3\hat{\imath} - 2\hat{\jmath}$ and $f(\hat{\jmath}) = 2\hat{\imath} + \hat{\jmath}$, and the task is to calculate $f(5\hat{\imath} + 7\hat{\jmath})$.

$$\begin{aligned} f(5i+7j) &= 5f(i) + 7f(j) \quad \leftarrow \text{from defn. of lin. transf!} \\ &= 5(3i - 2j) + 7(2i + j) \\ &= \ldots \\ &= 29i - 3j \end{aligned}$$

also written $\begin{bmatrix} 29 \\ -3 \end{bmatrix}$ or just $\boxed{(29, -3)}$

The reason

$$\begin{bmatrix} 3 & 2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 7 \end{bmatrix} = \begin{bmatrix} 29 \\ -3 \end{bmatrix}$$

is calculated in this way is because it fits nicely with the geometric idea of applying a linear transformation that moves $(5, 7)$ to to $(29, -3)$.

Every linear transformation has a corresponding matrix, and every matrix describes a linear transformation.

For example, $\begin{bmatrix}3 & 2\\ -2 & 1\end{bmatrix}$ corresponds to the transformation

$$f(x, y) = (3x + 2y, \ -2x + y).$$

because

$$\begin{bmatrix}3 & 2\\ -2 & 1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} = \begin{bmatrix}3x + 2y\\ -2x + y\end{bmatrix}$$

or

$$\begin{bmatrix}3 & 2\\ -2 & 1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} = x\begin{bmatrix}3\\ -2\end{bmatrix} + y\begin{bmatrix}2\\ 1\end{bmatrix}.$$

This column is $f(\hat{\imath})$.

This column is $f(\hat{\jmath})$.

How should we calculate

$$\begin{bmatrix} 0 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 7 \end{bmatrix} ?$$

Option 1:

- First, do $\begin{bmatrix} 3 & 2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 7 \end{bmatrix} = \begin{bmatrix} 29 \\ -3 \end{bmatrix}$.

  This is applying g for which $g(\hat{\imath}) = (3, -2)$ and $g(\hat{\jmath}) = (2, 1)$.
- Then, do $\begin{bmatrix} 0 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 29 \\ -3 \end{bmatrix} = \begin{bmatrix} -6 \\ 75 \end{bmatrix}$.

  This is applying f for which $f(\hat{\imath}) = (0, 3)$ and $f(\hat{\jmath}) = (2, 4)$.

How should we calculate

$$\begin{bmatrix} 0 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 7 \end{bmatrix} ?$$

Option 2:

- First, do $\begin{bmatrix} 0 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ 1 & 10 \end{bmatrix}$.

  This is using a new function $f(g(\vec{v}))$.
- Then do $\begin{bmatrix} -4 & 2 \\ 1 & 10 \end{bmatrix} \begin{bmatrix} 5 \\ 7 \end{bmatrix} = \begin{bmatrix} -6 \\ 75 \end{bmatrix}$.

Summary

Matrices describe linear transformations, and every linear tr. has a matrix.

In 2D: any linear transformation $f : \mathbb{R}^2 \to \mathbb{R}^2$ will have the formula

$$f(x, y) = \begin{bmatrix} ax+by \\ cx+dy \end{bmatrix},$$

for some 4 numbers a, b, c, d. This is exactly the same as

$$f(x, y) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Summary

The way we calculate $A\vec{v}$ is defined to match with doing a linear transformation (moving a point $\vec{v}$ to its image).

The way we calculate AB is defined so that $(AB)\vec{v}$ will match doing the linear transformation for B and then the linear transformation for A.

- This is why $AB \neq BA$!
  Rotate-then-scale is not the same as scale-then-rotate.

It's easiest to think of all of this for $\mathbb{R}^2 \to \mathbb{R}^2$, but in fact this works for $\mathbb{R}^n \to \mathbb{R}^n$ and even for $\mathbb{R}^n \to \mathbb{R}^m$.

Warm-up 1: Solve the system

$$\begin{cases} 2x + 3y = 5 \\ 4x + 8y = 6 \end{cases}$$

STAND UP when you have the answer.

Warm-up 2: Multiply the matrices

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 32 \\ 7 \end{bmatrix}.$$

We can check that the formula

$$\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}1x+0y\\0x+1y\end{bmatrix}=\begin{bmatrix}x\\y\end{bmatrix}$$

is correct, but if we think of a matrix's columns as recording the transformed positions of $\hat{\imath}$ and $\hat{\jmath}$, then it's even easier to see that $\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$ doesn't actually move points at all:

$$\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$

where $\hat{\imath}$ lands

where $\hat{\jmath}$ lands

But this is the same as where $\hat{\imath}$ and $\hat{\jmath}$ started!

The **$n \times n$ identity matrix**, written $I_{n\times n}$ or I_n or just I, is a special matrix such that

- $I\vec{v} = \vec{v}$ for any vector $\vec{v}$,
- $IM = M$ for any matrix M,
- $MI = M$ for any matrix M

if the products exist.

In a way, each matrix $I_{n\times n}$ acts like the number 1 because multiplying by it does not cause any change.

Formulas: $I_{2\times2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. $I_{3\times3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. $I_{4\times4} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

Multiply $\begin{bmatrix} 8 & -3 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 4 & 8 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$

$$\frac{1}{4}\begin{bmatrix} 8 & -3 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 4 & 8 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 8/4 & -3/4 \\ -4/4 & 2/4 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 4 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -3/4 \\ -1 & 1/2 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 4 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Inverse

With numbers, we know $1x = x$ and we know $x \cdot x^{-1} = x \cdot \frac{1}{x} = 1$.

The **inverse** of matrix A, which we write as A^{-1}, is a matrix that satisfies

$$AA^{-1} = I \quad \text{and} \quad A^{-1}A = I.$$

- For example, we just showed that $\begin{bmatrix} 2 & 3 \\ 4 & 8 \end{bmatrix}^{-1} = \begin{bmatrix} 2 & -3/4 \\ -1 & 1/2 \end{bmatrix}$.

Inverse

With numbers, we know $1x = x$ and we know $x \cdot x^{-1} = x \cdot \frac{1}{x} = 1$.

The **inverse of matrix A**, which we write as A^{-1}, is a matrix that satisfies

$$AA^{-1} = I \quad \text{and} \quad A^{-1}A = I.$$

- Some matrices don't have inverses. We say they are **non-invertible**.
  - This is similar to 0 for numbers, but this can happen even for some matrices that don't have any 0s in them!
- If A^{-1} does exist, then we say A is **invertible**.
  - Any invertible matrix will be a square matrix (same # of row as cols).

How can we use the inverse of a matrix? Example: $\begin{bmatrix} 2 & 3 \\ 4 & 8 \end{bmatrix}^{-1} = \begin{bmatrix} 2 & -3/4 \\ -1 & 1/2 \end{bmatrix}$.

- Solving systems of equations.

Warm-up: Solve the system

$$\begin{cases} 2x + 3y = 5 \\ 4x + 8y = 6 \end{cases}$$

$$A\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

$$A^{-1}A\begin{bmatrix} x \\ y \end{bmatrix} = A^{-1}\begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

$$I\begin{bmatrix} x \\ y \end{bmatrix} = A^{-1}\begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = A^{-1}\begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & -3/4 \\ -1 & 1/2 \end{bmatrix}\begin{bmatrix} 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 11/2 \\ -2 \end{bmatrix}$$

How can we use the inverse of a matrix? Example: $\begin{bmatrix} 2 & 3 \\ 4 & 8 \end{bmatrix}^{-1} = \begin{bmatrix} 2 & -3/4 \\ -1 & 1/2 \end{bmatrix}$.

- Solving equations with matrices.
  Example: Find M such that $\begin{bmatrix} 4 & 0 \\ 2 & 5 \end{bmatrix} = M \begin{bmatrix} 2 & 3 \\ 4 & 8 \end{bmatrix}$.
- “Undoing” a linear transformation.
- Decrypting an message that was encrypted using a matrix algorithm.

There are many other uses if $\begin{bmatrix} 2 & 3 \\ 4 & 8 \end{bmatrix}$ appears in an engineering problem.

Determinant

For 2×2 matrices there is a simple formula for the inverse:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

- The number $ad - bc$ is called the **determinant** of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.
  - We write $\det(M)$ for the determinant of matrix M. Some textbooks write M , but determinants can be < 0.
  - The inverse A^{-1} exists if and only if the determinant of A is *not* 0.

For larger square matrices, $\begin{bmatrix} a & \cdots \\ \vdots & \ddots \end{bmatrix}^{-1} = \frac{1}{\det(M)} \begin{bmatrix} ?? & \cdots \\ \vdots & \ddots \end{bmatrix}$ in some way.

Determinant

For 2×2 matrices,

$$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc.$$

We can also write

$$\det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

or

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.$$

Next week: determinant of 3×3 matrix.